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REVISIONS OF CONSTRAINTS APPROXIMATION IN THE SUCCESSIVE QP METHOD FOR NONLINEAR PROGRAMMING PROBLEMS

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Abstract

In the last few years the successive quadratic programming methods proposed by Han and Powell have been widely recognized as excellent means for solving nonlinear programming problems.

However, there remain some questions about their linear approximations to the constraints from both theoretical and empirical points of view.

In this paper, we propose two revisions of the linear approximation to the constraints and show that the directions generated by the revisions are also descent directions of exact penalty functions of nonlinear programming problems. The new technique can cope better with bad starting points than the usual one.

Key words:

Nonlinear Programming, Successive Quadratic Programming Method, Linear Approximations, Start Procedure, Exact Penalty Function.

Abbreviated title:

Constraints Approximation for Nonlinear Programming.

1. *Introduction*

The nonlinear programming problem to be considered in this paper is defined as

$$(NLP) \qquad \min_{x} f(x), \tag{1.1}$$

subject to

$$c_i(x) = 0, \qquad (i \in I_1) \tag{1.2}$$

$$c_i(x) \geq 0, \qquad (i \in I_2) \tag{1.3}$$

where f, $c_i : R^n \to R$ and I_1 and I_2 are the index sets of equality and inequality constraints, respectively.

Powell [5] defines an associated quadratic programming problem with an approximation x to the solution as follows:

$$(QP(x,H)) \qquad \min_{p} \nabla f(x)^T p + \frac{1}{2} p^T H p \tag{1.4}$$

subject to

$$c_i(x) + \nabla c_i(x)^T p = 0, \quad (i \in I_1) \tag{1.5}$$

$$c_i(x) + \nabla c_i(x)^T p \geq 0, \quad (i \in I_2) \tag{1.6}$$

where the $n \times n$ symmetric matrix H is a positive definite approximation to the Hessian of the Lagrangian

$$L(x,\lambda) = f(x) + \lambda^T c(x). \tag{1.7}$$

(1.5) and (1.6) are linear approximations to the constraints (1.2) and (1.3). If they are inconsistent, Powell [5] introduces an extra variable s into the quadratic programming and replaces the constraints (1.5) and (1.6) by the conditions

$$c_i(x)s + \nabla c_i(x)^T p = 0, \qquad (i \in I_1) \tag{1.8}$$

$$c_i(x)s_i + \nabla c_i(x)^T p \geqq 0, \qquad (i \in I_2) \tag{1.9}$$

where s_i has the value

$$s_i = 1, \text{ if } c_i(x) > 0, \tag{1.10}$$

$$s_i = s, \text{ if } c_i(x) \leqq 0. \tag{1.11}$$

s should be made as large as possible subject to the condition

$$0 \leqq s \leqq 1 \tag{1.12}$$

and any freedom in p is used to minimize the objective function (1.4). This can be done by adding the penalty term $- Ms$ to(1.4).

If the modified quadratic program has the only feasible solution $s = 0$ and $p = 0$, the algorithm finishes because it is assumed that the constraints (1.2) and (1.3) are inconsistent. (The conclusion is true if all the constraints (1.2) and (1.3) are linear.) It is not difficult to find an example which is feasible originally but is judged to be inconsistent with respect to the modified linear approximation. From experience, we know that nonlinear equality constraints are often the cause of troubles. In order to overcome such inconsistency, we have to modify the QP subproblem, especially, the constaints approximation. In Sections 3 and 4, we propose two modified QP subproblems which are derived by relaxing the approximation. One puts emphasis on the relaxation of nonlinear equality constraints, while the other relaxes all the constraints in the subproblem. Both QP subproblems are always consistent and have wider feasible regions than does Powell's modification. We show the important property that the new directions generated by the revisions are descent directions of exact penalty functions of the nonlinear programming problem.

In a remark in Section 4, we compare our results with Biggs's recursive QP subproblem [1].

2. *A simple example*

We consider the following problem

(NLP 1) $\min f(x) \equiv x_1{}^3 + x_2{}^2,$

subject to

$$c_1(x_1, x_2) \equiv x_1{}^2 + x_2{}^2 - 10 = 0, \qquad (2.1)$$

$$c_2(x_1, x_2) \equiv x_1 - 1 \geqq 0, \qquad (2.2)$$

$$c_3(x_1, x_2) \equiv x_2 - 1 \geqq 0. \qquad (2.3)$$

This problem is feasible and has the optimal solution $x_1 = 1$, $x_2 = 3$.
Let a starting value be $x_1 = -10$ and $x_2 = -10$.
Then the approximations to the constraints corresponding to (1.2) and (1.3) are:

$$190 - 20p_1 - 20p_2 = 0,$$

$$-11 + p_1 \geqq 0,$$

$$-11 + p_2 \geqq 0.$$

It is easy to see that the system is inconsistent.
The modified system corresponding to (1.8) and (1.9) is:

$$190s - 20p_1 - 20p_2 = 0,$$

$$-11s + p_1 \geqq 0,$$

$$-11s + p_2 \geqq 0.$$

The system has the only solution $s = 0$, $p_1 = 0$, $p_2 = 0$, and the original problem will thus be assumed inconsistent even though it is feasible. This simple example shows that the modified approximation does not always work well and suggests the necessity for other ones. The main cause of such troubles is that the freedom of p becomes very restricted as we approximate the violated equality or inequality constraints by (1.8) or (1.9). A natural way to avoid this problem is to relax the constraints and in our case to introduce the extra slack variables similar to the linear programming case. For example, we relax (1.5) by introducing slack variables t_i^1 and t_i^2 as follows:

$$c_i(x) + \nabla c_i(x)^T p - t_i^1 + t_i^2 = 0, \qquad (t_i^1,\ t_i^2 \geqq 0)$$

where $t_i^1 + t_i^2$ should be made as small as possible.

Above all, the following are important factors to be considered in designing the modifications.

(1) The new direction p generated by the modified QP subproblem should be a descent direction of an exact penalty function of the nonlinear programming problem.

(2) Since bad starting values of x often cause the inconsistency, we should design the modifications in such a way that it is easy to go back to the original successive QP methods as soon as (1.5) and (1.6) recover the consistency at a suitable stage.

In the following sections, we propose two modified QP subproblems designed according to the above mentioned plan. The first one lays emphasis on the relaxation of the violated nonlinear equality constraints, while the second one is more general in relaxing the constraints but needs the introduction of more slack variables than the first one.

3. *The first modified QP subproblem*

As was mentioned above, the first modification puts emphasis on the resolution of the inconsistency caused by the nonlinear equality constraints.

In the case that the approximations (1.2) and (1.3) are found to be inconsistent, we revise them in the following way.

$$(MQP1(\overline{x,H})) \quad \min_{p,t^1,t^2,s} \nabla f(x)^{\mathrm{T}} p + \frac{1}{2} p^{\mathrm{T}} H p + M_1 \sum_i (t_i^1 + t_i^2) - M_2 s, \qquad (3.1)$$

subject to

$$c_i(x) + \nabla c_i(x)^{\mathrm{T}} p - t_i^1 + t_i^2 = 0, \quad (i \in \mathrm{I}_{11} \cup \mathrm{I}_{13}) \qquad (3.2)$$

$$\nabla c_i(x)^{\mathrm{T}} p = 0, \quad (i \in \mathrm{I}_{12}) \qquad (3.3)$$

$$c_i(x) + \nabla c_i(x)^{\mathrm{T}} p \geqq 0, \quad (i \in \mathrm{I}_{21} \cup \mathrm{I}_{22}) \qquad (3.4)$$

$$c_i(x) s + \nabla c_i(x)^{\mathrm{T}} p \geqq 0, \quad (i \in \mathrm{I}_{23}) \qquad (3.5)$$

$$0 \leqq s \leqq 1, \qquad (3.6)$$

$$t_i^1,\ t_i^2 \geqq 0, \quad (i \in \mathrm{I}_{11} \cup \mathrm{I}_{13})$$

where M_1 and M_2 are sufficiently large positive numbers, and

$$\begin{aligned} \mathrm{I}_{k1} &= \{i \mid i \in \mathrm{I}_k,\ c_i(x) > 0\}, \\ \mathrm{I}_{k2} &= \{i \mid i \in \mathrm{I}_k,\ c_i(x) = 0\}, \\ \mathrm{I}_{k3} &= \{i \mid i \in \mathrm{I}_k,\ c_i(x) < 0\}. \end{aligned} \qquad (3.7)$$

$$(k = 1,\ 2)$$

It is easy to see that $MQP1(x,H)$ is feasible if the system (1.8) - (1.12) is consistent. In this sense, $MQP1(x,H)$ has a wider range of applications to solve nonlinear programming problems. Also, since it has an obvious solution $p=0$, $s=0$, $t_i^1 = \max\{0, c_i(x)\}$ and $t_i^2 = \max\{0, -c_i(x)\}$, it is always consistent.

An algorithm based on this modification goes as follows:

Let a Kuhn-Tucker solution to $MQP1(x,H)$ be p, t^1, t^2 and s. Then, if $p = 0$, the constraints (1.2) and (1.3) are assumed to be inconsistent or the approximation x is not well suited for the iteration. Otherwise, the direction p will be used to find out the next approximation $\bar{x} = x + \alpha p$ $(0<\alpha \leq 1)$ to the solution of the nonlinear programming problem by minimizing the exact penalty function which will be defined in (3.8).

We will not discuss the line search strategies and the updating formulae of the matrix H at each iteration, since our modifications are on the extensions of Powell's and Han's methods and we can use several strategies and updating formulae proposed in [2] and [5].

Next, we will show the descent property of the direction p. Let the Kuhn-Tucker dual solutions corresponding to (3.2), (3.3), (3.4), (3.5) and (3.6) be u_i, v_i, w_i, y_i and z, respectively. Then, we can demonstrate that the direction p is a descent direction of the exact penalty functions θ,

$$\theta(x) = f(x) + r_1 \sum_{i\varepsilon I_1} |c_i(x)| - r_2 \sum_{i\varepsilon I_2} \min\{0, c_i(x)\}, \qquad (3.8)$$

where r_1 and r_2 are positive numbers.

Theorem 3.1

Let f and c_i ($i \in I_1 \cup I_2$) be continuously differentiable at x and H be a positive definite $n \times n$ symmetric matrix.

If $(p, t^1, t^2, s, u, v, w, y, z)$ is a Kuhn-Tucker solution of MQP1 (x,H) with $p \neq 0$ and

$$r_1 = M_1 \tag{3.9}$$

$$r_2 \geqq \|y\|_\infty, \tag{3.10}$$

then p is a descent direction of θ at x.

Proof:

The directional derivative $D_p\theta$ of θ at x along the direction p is given by

$$\begin{aligned} D_p\theta(x) = \nabla f(x)^T p &+ r_1\{\sum_{i\in I_{11}} \nabla c_i(x)^T p - \sum_{i\in I_{13}} \nabla c_i(x)^T p \\ &+ \sum_{i\in I_{12}} |\nabla c_i(x)^T p|\} - r_2\{\sum_{i\in I_{23}} \nabla c_i(x)^T p \\ &+ \sum_{i\in I_{22}} \min\{0, \nabla c_i(x)^T p\}\} \end{aligned} \tag{3.11}$$

(Refer to Dem'yanov and Malozemov [3] or Han [4]).

Since $(p, t^1, t^2, s, u, v, w, y, z)$ is a Kuhn-Tucker solution of $MQP1(x,H)$, we have

$$\begin{aligned} \nabla f + H p - \sum_{i\in I_{11}\cup I_{13}} u_i \nabla c_i(x) - \sum_{i\in I_{12}} v_i \nabla c_i(x) \\ - \sum_{i\in I_{21}\cup I_{22}} w_i \nabla c_i(x) - \sum_{i\in I_{23}} y_i \nabla c_i(x) \\ = 0, \end{aligned} \tag{3.12}$$

$$M_1 \geqq |u_i|, \qquad (i \in I_{11} \cup I_{13}) \tag{3.13}$$

$$w \geqq 0, \; y \geqq 0, \; z \geqq 0, \tag{3.14}$$

$$v_i \nabla c_i(x)^T p = 0, \qquad (i \in I_{12}) \tag{3.15}$$

$$w_i(c_i(x) + \nabla c_i(x)^T p) = 0, \quad (i \in I_{21} \cup I_{22}) \tag{3.16}$$

$$y_i(c_i(x)s + \nabla c_i(x)^T p) = 0, \quad (i \in I_{23}) \tag{3.17}$$

$$t_i^1(M_1 + u_i) = 0, \qquad (i \in I_{11} \cup I_{13}) \tag{3.18}$$

$$t_i^2(M_1 - u_i) = 0. \qquad (i \in I_{11} \cup I_{13}) \tag{3.19}$$

From (3.11), (3,12) and (3.15) - (3.17), we get

$$\begin{aligned} D_p\theta(x) = & -p^T H p + \sum_{i \in I_{11}} r_1 \nabla c_i(x)^T p - \sum_{i \in I_{13}} r_1 \nabla c_i(x)^T p \\ & + \sum_{i \in I_{12}} r_1 |\nabla c_i(x)^T p| - \sum_{i \in I_{11} \cup I_{13}} u_i \nabla c_i(x)^T p \\ & - \sum_{i \in I_{23}} r_2 \nabla c_i(x)^T p - \sum_{i \in I_{22}} r_2 \min\{0, \nabla c_i(x)^T p\} - \sum_{i \in I_{21} \cup I_{22}} w_i c_i(x) \\ & - \sum_{i \in I_{23}} y_i s c_i(x). \end{aligned} \tag{3.20}$$

(3.3) and (3.4) mean that

$$|\nabla c_i(x)^T p| = 0, \qquad (i \in I_{12}) \tag{3.21}$$

$$\min\{0, \nabla c_i(x)^T p\} = 0. \qquad (i \in I_{22}). \tag{3.22}$$

From (3.20) - (3.22), (3.2) - (3.5) and (3.9), we have

$$D_p\theta(x) \leqq -p^T H p + \sum_{i\in I_{11}} (M_1+u_i)\nabla c_i(x)^T p + \sum_{i\in I_{13}} (-M_1+u_i)\nabla c_i(x)^T p$$

$$+ s\cdot\sum_{i\in I_{23}} (r_2 - y_i)\, c_i(x). \tag{3.23}$$

(3.2), (3.18) and (3.19) mean that

if $i\in I_{11}$ and $\nabla c_i(x)^T p > 0$, then

$$t_i^1 = c_i(x) + \nabla c_i(x)^T p > 0,\ t_i^2 = 0,\ M_1 + u_i = 0 \tag{3.24}$$

and

if $i\in I_{13}$ and $\nabla c_i(x)^T p < 0$, then

$$t_i^2 = -c_i(x) - \nabla c_i(x)^T p > 0,\ t_i^1 = 0,\ M_1 - u_i = 0. \tag{3.25}$$

From (3.24), (3.25), (3.10) and (3.13), it follows that

$$(M_1 + u_i)\nabla c_i(x)^T p \leqq 0, \qquad (i \in I_{11})$$

$$(-M_1 + u_i)\nabla c_i(x)^T p \leqq 0, \qquad (i \in I_{13})$$

$$s\sum_i (r_2 - y_i)\, c_i(x) \leqq 0. \qquad (i \in I_{23})$$

Therefore, we can conclude

$$D_p\theta(x) \leqq -p^T Hp < 0.$$

By applying the modification to the sample problem in Section 2, the corresponding $MQP1(x,H)$ has the optimal solution with $p_1 = 11$, $p_2 = 11$, $t_1^1 = 0$, $t_1^2 = 250$ and $s = 1$. The next iteration starts from $\bar{x}_1 = x_1 + p_1 = 1$ and $\bar{x}_2 = x_2 + p_2 = 1$ and after five iterations we have the optimal solution $x_1 = 1$ and $x_2 = 3$, with no further recourse to *MQP1*.

Remark 3.1 Since the role of the variable s is not particularly significant (although it should be as large as possible from the point of view of approximation) and the coefficient M_2 of s in the QP objective function has no explicit relation with the penalty parameter r_2 and since nonlinear equality constraints often cause inconsistency in the constraints approximations, it may be sufficient to put $s = 1$ and introduce only $\{t_i^1, t_i^2\}$ in $MQP1(x,H)$.

Remark 3.2 The direction p will be used to find the next point $\bar{x} = x + \alpha p$ $(0 < \alpha \leqq 1)$ by minimizing the penalty function θ or by minimizing some approximation to the Lagrangian function on this line.

To guarantee convergence to the required solution of the nonlinear programming problem, the penalty parameter r_1 and r_2 in $\theta(x)$ should be sufficiently large and indeed be as large as $r_1 = M_1$ and $r_2 \geqq \| y \|_\infty$, where y is the Lagrange multiplier vector for the violated inequality constraints on each iteration of QP subproblems. Several methods proposed in [2] with respect to line search and updating of the matrix H, may be usefull in our case, too.

4. *The second modified QP subproblem*

The second problem has a wider feasible region than *MQP1* but needs more extra slack variables.

$$(\mathit{MQP2}(\bar{x}, \bar{H})) \qquad \min_{p,t^1,t^2,s} \quad \nabla f(x)^T p + \tfrac{1}{2} p^T H p + \sum_{i \in I_1} M_i^1 (t_i^1 + t_i^2) + \sum_{i \in I_2} M_i^2 s_i, \tag{4.1}$$

subject to

$$c_i(x) + \nabla c_i(x)^T p - t_i^1 + t_i^2 = 0, \quad (i \in I_1) \tag{4.2}$$

$$c_i(x) + \nabla c_i(x)^T p + s_i \geqq 0, \qquad (i \in I_2) \tag{4.3}$$

$$t_i^1, \ t_i^2, \ s_i \geqq 0, \tag{4.4}$$

where M_i^1 and M_i^2 are sufficiently large positive numbers.
Let a Kuhn-Tucker solution be p, t^1, t^2 and s. Then, we can demonstrate the descent property of the direction p for the exact penalty function ψ,

$$\psi(x) = f(x) + \sum_{i \in I_1} r_i^1 |c_i(x)| - \sum_{i \in I_2} r_i^2 \min\{0, c_i(x)\} \tag{4.5}$$

where r_i^1 and r_i^2 are positive numbers.

Theorem 4.1

Let f and c_i ($i \in I_1 \cup I_2$) be continuously differentiable at x and H be a positive definite $n \times n$ symmetric matrix.

If (p, t^1, t^2, s) is a Kuhn-Tucker solution of $MQP2(x,H)$ with $p \neq 0$ and

$$r_i^1 = M_i^1, \qquad (i \in I_1) \tag{4.6}$$

$$r_i^2 = M_i^2, \qquad (i \in I_2) \tag{4.7}$$

then p is a descent directions of ψ at x.

Proof:

Let the Kuhn-Tucker dual solutions corresponding to (4.2) and (4.3) be u_i and w_i, respectively.

The directional derivative $D_p \psi$ of ψ at x along the direction p is given by

$$\begin{aligned} D_p\psi(x) = {} & \nabla f(x)^T p + \sum_{i \in I_{11}} r_i^1 \nabla c_i(x)^T p - \sum_{i \in I_{13}} r_i^1 \nabla c_i(x)^T p \\ & + \sum_{i \in I_{12}} r_i^1 |\nabla c_i(x)^T p| - \sum_{i \in I_{23}} r_i^2 \nabla c_i(x)^T p \\ & + \sum_{i \in I_{22}} r_i^2 \min\{0, \nabla c_i(x)^T p\}. \end{aligned} \tag{4.8}$$

By using Kuhn-Tucker optimality relations, we have

$$\begin{aligned} D_p\psi(x) = {} & -p^T H p + \sum_{i \in I_{11}} (u_i + r_i^1) \nabla c_i(x)^T p \\ & + \sum_{i \in I_{13}} (u_i - r_i^1) \nabla c_i(x)^T p + \sum_{i \in I_{12}} \{u_i \nabla c_i(x)^T p + r_i^1 |\nabla c_i(x)^T p|\} \\ & + \sum_{i \in I_{21}} w_i \nabla c_i(x)^T p + \sum_{i \in I_{23}} (w_i - r_i^2) \nabla c_i(x)^T p \\ & + \sum_{i \in I_{22}} [w_i \nabla c_i(x)^T p + r_i^2 \min\{0, \nabla c_i(x)^T p\}]. \end{aligned} \tag{4.9}$$

From the optimality conditions, it follows that

(i) if $i \in I_{11} \cup I_{12}$ and $\nabla c_i(x)^T p > 0$, then

$$t_i^1 = c_i(x) + \nabla c_i(x)^T p > 0, \quad t_i^2 = 0 \text{ and } \bar{u}_i = -M_i^1,$$

(ii) if $i \in I_{13} \cup I_{12}$ and $\nabla c_i(x)^T p < 0$, then

$$t_i^2 = -c_i(x) - \nabla c_i(x)^T p > 0, \quad t_i^1 = 0 \text{ and } u_i = M_i^1,$$

(iii) if $i \in I_{21} \cup I_{22}$ and $\nabla c_i(x)^T p > 0$, then

$$s_i = 0 \text{ and } w_i = 0,$$

and

(iv) if $i \in I_{23} \cup I_{22}$ and $\nabla c_i(x)^T p < 0$, then

$$s_i = -c_i(x) - \nabla c_i(x)^T p > 0 \text{ and } w_i = M_i^2.$$

From the above relations, (4.6) and (4.7), we obtain

$$\begin{aligned} D_p\psi(x) &\leqq -p^T Hp + \sum_{i \in I_{11-}} (u_i + M_i^1)\nabla c_i(x)^T p \\ &+ \sum_{i \in I_{13+}} (u_i - M_i^1)\nabla c_i(x)^T p \\ &+ \sum_{i \in I_{23+}} (w_i - M_i^2)\nabla c_i(x)^T p \\ &\leqq -p^T H p < 0, \end{aligned}$$

where $I_{\alpha\beta}+$ and $I_{\alpha\beta}-$ denote the sets $\{i \mid i \in I_{\alpha\beta}$ and $\nabla c_i(x)^T p > 0\}$ and $\{i \mid i \in I_{\alpha\beta}$ and $\nabla c_i(x)^T p < 0\}$, respectively. The second relation comes from the optimality conditions $M_i^1 \geqq |u_i|$ $(i \in I_1)$ and $M_i^2 \geqq w_i$ $(i \in I_2)$.

Remark 4.1 Biggs [1] proposes a recursive QP subproblem for which constraints are of the form:

$$c_i(x) + \nabla c_i(x)^T p + \frac{1}{2}\bar{r}\,\bar{u}_i = 0, \quad (i \in I_1) \tag{4.10}$$

$$c_i(x) + \nabla c_i(x)^T p + \frac{1}{2}\bar{r}\,\bar{w}_i \geqq 0, \quad (i \in I_2) \tag{4.11}$$

where the $\bar{u}_i$ and $\bar{w}_i$ are estimated Lagrange multipliers and $\bar{r}$ is a penalty parameter. It is shown that $\bar{u}_i$ and $\bar{w}_i$ can be defined in such a way that these constraints are always consistent for positive $\bar{r}$. (4.10) and (4.11) are linearlized and *perturbed* forms of the original constraints which assure the consistency, while $MQP2(x,H)$ introduces the perturbations t_i^1, t_i^2 and s_i as the solutions only when they are necessary. Let a feasible solution of (4.10) and (4.11) be p_B. Then, $p = p_B$, $t_i^1 = \max\{0, -\frac{1}{2}\bar{r}\,\bar{u}_i\}$, $t_i^2 = \max\{0, \frac{1}{2}\bar{r}\,\bar{u}_i\}$ and $s_i = \frac{1}{2}\bar{r}\,\bar{w}_i$ are a feasible solution of $MQP2(x,H)$. Thus, $MQP\ 2(x,H)$ contains the optimal direction of Biggs's sub-problem. It is very interesting that both formulations resemble each other, although they come from different origins. Comparison of both methods with respect to line search strategies, updating of the matrix H, computational efficiency and robustness against bad starting points, is a subject of future research and experimentation.

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